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A Method for Deriving Probability Distributions with Gamma Functions

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PREFACE

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Based on an exponential integral formula recently derived by the authors, a substitution method is presented for determining a probability density function and moments for a useful class of $[0,\infty)$ exponentials frequently encountered in signal processing. The generalization of the method is elaborated to account for models of $(-\infty,\infty)$ exponentials. The generalized density function and moments relation are given as integral formulas specifically calculated from their definitions in probability theory. Several examples of the method are provided, including an example for modeling random signals and deterministic acoustic signals in underwater acoustics. The generalization of the method for multivariate distributions is presented.

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CHANGE TO TECHNICAL REPORT

To all holders of
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1. NUWC-NPT TR 10,474 has been changed to correct several typographical errors.

2. Please make the following pen-and-ink changes:

a. On page 11, table 2, change the second expression in the PDF column from

$$\frac{\lambda^{\mu}}{\Gamma(\mu)} x^{\mu} e^{-\lambda x} \quad \text{to} \quad \frac{\lambda^{\mu}}{\Gamma(\mu)} x^{\mu-1} e^{-\lambda x}.$$

b. On page 13, change the first line of equation (32) from

$$E(\mathbf{x}) = \frac{a^{-1/2}\Gamma(2)}{\sqrt{\pi / 2}} \quad \text{to} \quad E(\mathbf{x}) = \frac{a^{-1/2}\Gamma(2)}{\sqrt{\pi}/2}.$$

c. On page 15, change the first equation from

$$E(\mathbf{x}) = \beta^{-1/2} \frac{\Gamma\left(\frac{0+1+1}{2}\right)}{\left(\frac{0+1}{2}\right)} = a^{-1/2} \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{a\pi}}, \qquad \text{from (16)}$$

to

$$E(\mathbf{x}) = \beta^{-1/2} \frac{\Gamma\left(\frac{0+1+1}{2}\right)}{\Gamma\left(\frac{0+1}{2}\right)} = a^{-1/2} \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{a\pi}}, \qquad \text{from (16).}$$

d. On page 15, change the second equation from

$$E(\mathbf{x}) = \beta^{-2/2} \frac{\Gamma\left(\frac{0+2+1}{2}\right)}{\left(\frac{0+1}{2}\right)} = \frac{a^{-1}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{2a}, \qquad \text{from (18)}$$

to

$$E\left(\mathbf{x}^2\right) = \beta^{-2/2}\frac{\Gamma\left(\frac{0+2+1}{2}\right)}{\Gamma\left(\frac{0+1}{2}\right)} = \frac{a^{-1}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{2a}, \qquad \text{from (18).}$$

e. On page 15, change the third equation from the bottom of the page from

$$E(\mathbf{x}) = \beta^{1/2}\frac{\Gamma\frac{m+1+1}{n}}{\Gamma\left(\frac{m+1}{n}\right)} = \frac{1}{2\sqrt{\frac{\pi}{a}}}, \qquad \text{from (16)}$$

to

$$E(\mathbf{x}) = \beta^{-1/2}\frac{\Gamma\left(\frac{m+1+1}{n}\right)}{\Gamma\left(\frac{m+1}{n}\right)} = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}, \qquad \text{from (16).}$$

f. On page 18, change the third equation from

$$\frac{\partial\left(\sum_{i=1}^{p}\left[\log y_1 - \log\alpha - m\log x_i - \beta x_i^n\right]^2\right)}{\partial n}$$

to

$$\frac{\partial\left(\sum_{i=1}^{p}\left[\log y_i - \log\alpha - m\log x_i + \beta x_i^n\right]^2\right)}{\partial n}.$$

g. On page C-1, change “Reference 4” to “Reference 5” in the third line of the first paragraph.

h. On page C-1, change the second sentence under “NOTATION” to read “Let each occurrence of a random variable **x** now be denoted $\mathbf{x}_i$ and let....”

i. On page C-3, change the first equation from

$$f\left(x_1, x_2, \ldots, x_n\right) = \prod_{i=1}^{n}\frac{n_i b_i^{g_i} x_i^{m_i}}{\Gamma\left(\gamma_i\right)}e^{-b_i x_i^{n_i}} \qquad \text{to} \qquad f\left(x_1, x_2, \ldots, x_n\right) = \prod_{i=1}^{n}\frac{n_i \beta_i^{r_i} x_i^{m_i} e^{-\beta_i x_i^{n_i}}}{\Gamma\left(\gamma_i\right)}.$$

j. On page C-3, delete the fourth equation from the bottom of the page.

k. On page C-4, change the third equation from

$$E(\mathbf{y}^j) = 6^{-j} j^2 \Gamma(j) \quad \text{to} \quad E(\mathbf{y}^j) = 6^{-j} j^2 \Gamma(j)\, \Gamma(j).$$

24 June 1996

P. A. La Brecque
Head, Combat Systems Department

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A METHOD FOR DERIVING PROBABILITY DISTRIBUTIONS WITH GAMMA FUNCTIONS

INTRODUCTION

Of the many continuous probability distributions encountered in signal processing, a good number are distinguished by the fact that they are derived from exponential functions on $[0,\infty)$, the normal (Gaussian) distribution being a notable exception; e.g., failure-rate distributions, Poisson processes, chi-square, gamma, exponential, Rayleigh, Weibull, and others involving exponential functions. The normal distribution is a hyperbolic function on $(-\infty,+\infty)$ requiring separate consideration. In signal processing, the exponential function is a common choice to model correlated data structures such as correlated measurement noise (reference 1). Signal characterization stochastic randomness models also include exponential functions (see reference 2).

Occasionally, modeling involves functions for which the probability density function (PDF) and its moments need to be derived *de novo*. Often, research scientists and engineers are confronted with modeling a random variable x when the PDF is unknown. It may be known that the variable can reasonably be approximated by a gamma density. Then, solving a problem under the assumption that x has a gamma density will provide some insight into the true situation. This suggestion (from reference 3, chapter 5) is all the more reasonable because many probability distributions are related to the gamma function.

This report offers a tutorial approach for deriving a PDF and moments for a certain class of $[0,\infty)$ exponential functions based on a recently derived exponential integral formula. Reference 4 shows how the present method extends to $(-\infty,\infty)$. This work is an extension of Naval Undersea Warfare Center Division, Newport, RI, Technical Report 10,412 (reference 2). The standard and elegant approach to finding moments involving moment-generating functions and complex-variable characteristic functions is not being challenged. Rather, a method is introduced that is based on applied engineering mathematics and is both practical and easier to implement in applied research settings. Freund (reference 5, p. 155) makes it clear that statisticians advocate the most straightforward approach.

A general exponential integral formula has been derived that will simplify the mathematics involved in deriving PDFs and moments for a useful class of continuous functions. Consider the following exponential integral formula of order n, where a, m, $n \neq 0$, $\beta > 0$:

$$\int_0^\infty \alpha x^m e^{-\beta x^n} dx = \alpha \frac{\Gamma(\gamma)}{n\beta^\gamma}, \qquad 0 < x < \infty , \tag{1}$$

where $\gamma = \dfrac{m+1}{n} > 0$, and $\Gamma(\bullet)$ represents the gamma function that is discussed below. The derivation of equation (1) was presented in references 6 and 7. Appendix A summarizes the derivation. Essentially, equation (1) is a generalization of known integral formulas. Equation (1) is the basic reduction formula required to find PDFs and moments for a large class of univariate distributions.* The authors refer to this formula as the Moi formula.†

Table 1 lists several frequently encountered continuous PDFs taken from Hoel (reference 3). Each of those densities can be expressed in terms of the α, m, n, β, γ components of the Moi formula. To construct any of the densities in table 1, one can write down its Moi formula equivalent. For example, the exponential density is $\lambda e^{-\lambda x},\ \ 0 < x < \infty$. Each of the densities in table 1 is distinguished by the fact that when integrated over the interval 0 to ∞, each is equal to 1, which is the definition of a PDF. Later, the method will be justified by deriving the general moments formula for exponential functions like those in table 1.

The use of the proposed method will be justified by application to functions with known solutions obtained by the traditional methods of moment-generating or characteristic functions. The equivalence between a moment generating function approach to finding moments and the current approach will be demonstrated. In addition, hypothetical functions are introduced and the entire method applied to the functions to exemplify the details of the method. Note that the general case, $-\infty < x < \infty$, has been solved (see reference 4), although this study concentrates on $[0,\infty)$ models.

* See appendix C for multivariate generalization.

† The theory of exponentials is not pursued, but the Moi formula of equation (1) applies to integrals of the form $I = \int_{-\infty}^{+\infty} \alpha x^{m} e^{-\beta x^{n}} dx$, such that, if the integrand is an integrable *odd function*, $h(-x) = -h(x)$, then I = 0 (e.g., $\int_{-\infty}^{+\infty} x e^{-\frac{1}{2}x^2} dx = 0$), and if the integrand is an integrable *even function*, or symmetrical, about x = 0, $h(x) = h(-x)$, then $I = \int_{-\infty}^{+\infty} \alpha x^{m} e^{-\beta x^{n}} dx = 2\int_{0}^{+\infty} \alpha x^{m} e^{-\beta x^{n}} dx = 2\dfrac{\alpha\Gamma(\gamma)}{n\beta^{\gamma}}$;

e.g., $\int_{-\infty}^{+\infty} x^2 e^{-x^2/2} dx = \sqrt{2\pi}$ (reference 4).

Table 1. Univariate Densities Based on Exponential Functions

	Moi Components				
Density*	α	m	n	β	γ
Exponential	λ	0	1	λ	1
Gamma	$\frac{\lambda^{\mu}}{\Gamma(\mu)}$	$\mu-1$	1	λ	μ
Chi-Square	$\frac{1}{\Gamma\left(\frac{\nu}{2}\right)2^{\frac{\nu}{2}}}$	$\frac{\nu}{2}-1$	1	$\frac{1}{2}$	$\frac{\nu}{2}$
Rayleigh	$2a$	1	2	a	1
Gamma-Poisson†	$\frac{d(c\lambda)^{m}}{\Gamma(m)}$	md-1	d	$c\lambda$	m
Weibull	ab	b-1	b	a	1
Maxwell	$\sqrt{2/\pi}$	2	2	$\frac{1}{2}$	$\frac{3}{2}$

Note: Densities selected from reference 3.
* To construct density functions, substitute the density components into the exponential function $\alpha x^{m} e^{-\beta x^{n}}$. Assume that x is the random variable for each distribution.
† Derived in reference 2.

PROBABILITY DENSITY FUNCTIONS AND MOMENTS

GAMMA FUNCTION

Because the integral in equation (1) involves the gamma function, a brief review of the properties of gamma functions will be given for reference later in this report.

The gamma function (or Euler's integral) is defined by the following improper integral:

$$\Gamma(x;\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx, \quad \alpha>0\,. \tag{2}$$

Equation (2) is sometimes written $\Gamma(\alpha)$ when x is understood. Equation (2) arises in many applications of probability theory in signal processing. The drawn curve of equation (2) when x is positive is a U-shaped graph depicting a continuous function.*

The properties of equation (2) are summarized as follows (for proofs see references 3, 5, 8, and 9).

1. $\Gamma(n)=(n-1)!$, if n is a positive integer. (3)

For example, $\Gamma(4)=3!=6$.

2. $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$, for any $\alpha>0$. (4)

For example, $\Gamma\left(\frac{5}{2}\right)=\Gamma\left(\frac{3}{2}+1\right)=\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$.

3. $\Gamma(n+1)=n!$, if n is a positive integer. (5)

For example, $\Gamma(4+1)=4!=24$.

4. $\Gamma\left(\frac{n}{2}\right)=\dfrac{(n-1)!\sqrt{\pi}}{2^{n-1}\left(\frac{n-1}{2}\right)!}$, if n is an odd positive integer. (6)

---

* Although $\Gamma(\alpha)$ can be computed for $\alpha<0$ (see reference 9), the restriction in equation (2) is necessary for reasons that will become apparent subsequently.

For example,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi};$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \tag{7}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}, \text{ etc.}$$

The relation $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ is a famous relation and has been proved in various ways. An independent proof based on the Moi formula of equation (1) is given in appendix B.

For other cases of $\alpha > 0$, numerical integration must be used to obtain the value of the gamma function. For example, $\Gamma\left(\frac{1}{3}\right) = 2.67893\ldots$ must be calculated by some quadrature method. See reference 10 for extensive tables and useful algorithms.

DENSITY FUNCTIONS

The first use of equation (1) occurs when one desires to find a one-dimensional PDF for an exponential function that the researchers have decided models their data satisfactorily.*

Let that function be denoted by

$$g(x) = \alpha x^m e^{-\beta x^n}, \ 0 < x < \infty, \tag{8}$$

where $\alpha, \beta > 0$, m, n are any real constants and $\gamma = n^{-1}(m+1) > 0$. For example, if the real-valued function $\frac{\lambda^\mu}{\Gamma(\mu)} x^{\mu-1} e^{-\lambda x}$ conforms to the conditions of equation (8), then

$$\alpha = \frac{\lambda^\mu}{\Gamma(\mu)}, m = \mu - 1, \beta = \lambda, n = 1, \text{ and } \gamma = \mu.$$

* The multivariate generalization of the method is contained in appendix C.

Finding a formula that determines the PDF for any function conforming to equation (8) involves transforming (8) so that

$$c\int_0^\infty g(x)dx = 1, \tag{9}$$

where c is the normalizing constant.

Substituting $g(x)$ from equation (8), equation (9) can be written as

$$c\int_0^\infty \alpha x^m e^{-\beta x^n} dx = 1 \ . \tag{10}$$

Because the solution to the integral in equation (10) was given in equation (1), then c is found to equal

$$c\int_0^\infty g(x)dx = c\left(\frac{\alpha\Gamma(\gamma)}{n\beta^\gamma}\right) = 1 \Rightarrow c = \left(\frac{n\beta^\gamma}{\alpha\Gamma(\gamma)}\right) \ , \tag{11}$$

where $\gamma = \dfrac{m+1}{n} > 0$. In the case of $g(x)$, $-\infty < x < \infty$, $c = \dfrac{1}{2}\left(\dfrac{n\beta^\gamma}{\alpha\Gamma(\gamma)}\right)$. Then, for any function corresponding to equation (8), the PDF $f(x)$ is given by combining equations (8) and (11):

$$f(x) = c\int_0^\infty g(x)dx = \frac{n\beta^\gamma}{\alpha\Gamma(\gamma)}\int_0^\infty g(x)dx \,,$$

$$f(x) = \frac{n\beta^\gamma}{\Gamma(\gamma)} x^m e^{-\beta x^n} \,,$$

$$f(x) \geq 0 \tag{12}$$

$$\int_0^\infty f(x)\ dx = 1 \,,$$

where $f(x)$ will denote the PDF of an arbitrary distribution. The Moi density of equation (12) can be verified by integrating; the result will be 1. Appendix C treats the multivariate case.

To give an example of equation (12) for the densities of table 1, the PDF for the exponential function, constructed from the components of equation (12) is

$$f(x) = \frac{1(\lambda)^1 x^0 e^{-\lambda x}}{\Gamma(1)} = \lambda e^{-\lambda x}.$$

MOMENTS

The moments of a PDF are important for several reasons. The first moment corresponds to the mean of the distribution, and the second moment allows a calculation of the dispersion or variance of the distribution. The mean and variance may then be used in the central limit theorem or normal approximation formula for purposes of hypothesis testing.

For an arbitrary moment of the class of functions being considered (where integration is restricted to the interval 0 to ∞) the jth moment is defined as:

$$E(\mathbf{x}^j) = \int_0^\infty x^j c g(x) dx = \int_0^\infty x^j f(x) dx \, , j > 0 \; , \tag{13}$$

where $f(x)$ is the assumed PDF of equation (12).

Expressed in terms of the normalized Moi density of equation (12), equation (13) is

$$E(\mathbf{x}^j) = \int_0^\infty x^j \frac{n\beta^\gamma x^m}{\Gamma(\gamma)} e^{-\beta x^n} dx \; ,$$

$$E(\mathbf{x}^j) = \int_0^\infty \frac{n\beta^\gamma x^{m+j}}{\Gamma(\gamma)} e^{-\beta x^n} dx \; . \tag{14}$$

Equation (14) is evaluated by the Moi formula of equation (1). Let $\alpha = \frac{n\beta^\gamma}{\Gamma(\gamma)}$, substitute $m + j$ for the parameter m, and $\gamma + j/n$ for the parameter γ in equation (1) to obtain

$$E(\mathbf{x}^j) = \beta^{-j/n} \frac{\Gamma\left(\gamma + \frac{j}{n}\right)}{\Gamma(\gamma)} = \beta^{-j/n} \frac{\Gamma\left(\frac{m+j+1}{n}\right)}{\Gamma\left(\frac{m+1}{n}\right)} \; . \tag{15}$$

Note that equation (15) provides the moments relation for the general case, $-\infty < x < \infty$. Equation (15) is a closed form solution for calculating any moment of a PDF conforming to the class of exponential functions under consideration. See appendix C for the n-variable case.

MEAN AND VARIANCE FOR ANY ARBITRARY PDF

The mean and variance for any arbitrary PDF $f(x)$ conforming to equation (12), can now be found. The mean, or first moment, is given by setting j to 1 in equation (15). Thus, the mean is

$$E(\mathbf{x}) = \beta^{-1/n} \frac{\Gamma\left(\frac{m+2}{n}\right)}{\Gamma\left(\frac{m+1}{n}\right)} \quad . \tag{16}$$

The variance is defined by

$$\sigma^2 = E(\mathbf{x}^2) - [E(\mathbf{x})]^2 \ , \tag{17}$$

which requires knowledge of the second moment. $E(\mathbf{x}^2)$ is computed from equation (15) for $j = 2$; that is:

$$E(\mathbf{x}^2) = \beta^{-2/n} \frac{\Gamma\left(\frac{m+3}{n}\right)}{\Gamma\left(\frac{m+1}{n}\right)} \quad . \tag{18}$$

For example, the reader can verify that for the exponential density of table 1, the mean and variance are given by

$$E(\mathbf{x}) = \frac{1}{\lambda}; E(\mathbf{x}^2) = \frac{2}{\lambda^2}; \sigma^2 = \frac{1}{\lambda^2} \quad . \tag{19}$$

In general, the moments for all of the known densities in table 1 can be derived by substitution of the appropriate parameters into equation (15).

RELATION TO MOMENT GENERATING FUNCTION

To contrast the current approach for finding moments to that offered by the MGF,* consider the gamma density of table 1, given by

$$f(x) = \frac{\lambda^{\mu} x^{\mu-1}}{\Gamma(\mu)} e^{-\lambda x}, \ 0 < x < \infty \quad . \tag{20}$$

---

* Consideration of complex-variable characteristic functions is omitted.

Now, by definition, the MGF of the distribution of a random variable $\mathbf{x}$ on $[0,\infty)$ is given by the following expectation:

$$M_x(t) = Ee^{tx}$$

$$Ee^{tx} = \int_0^\infty e^{tx} f(x)dx = \int_0^\infty \frac{\lambda^\mu x^{\mu-1}}{\Gamma(\mu)} e^{-x(\lambda-t)}dx \text{ ,} 0 < x < \infty \text{ ,} \tag{21}$$

where t is real. Evaluating the integral in equation (21) directly by the Moi formula of equation (1) (which saves some effort) results in

$$M_x(t) = \left(\frac{\lambda}{\lambda - t}\right)^\mu . \tag{22}$$

Then, the mean of the PDF in equation (20) is defined to be the first derivative of the MGF evaluated at $t = 0$:

$$E(\mathbf{x}) = M_x'(0) = \left[\frac{dM_x(t)}{dt}\right]_{t=0} = \frac{\mu}{\lambda} . \tag{23}$$

The second and higher moments are given by analogous constructions.

The proposed method for finding the mean and variance, or higher moments, avoids both the integration and differentiation needed to compute the MGF and moments. For the gamma density of equation (20), the mean is found in one step by substituting the appropriate parameters into the Moi MGF of equation (16). When this is done, the result is

$$E(\mathbf{x}) = \lambda^{-1} \frac{\Gamma(\mu\text{-}1+2)}{\Gamma(\mu)} = \frac{\mu}{\lambda} \text{ ,} \tag{24}$$

which is equivalent to equation (23).

This example suggests that for special cases there is an equivalence between the MGF and the current approach. This can be demonstrated for a general PDF of order 1. The PDF via Moi based on equation (12) is

$$f(x) = \frac{\beta^{m+1} x^m}{\Gamma(m+1)} e^{-\beta x} \text{ ,} \tag{25}$$

and the mean from the Moi moment generating function of equation (16) is

$$E(\mathbf{x}) = \beta^{-1}\frac{\Gamma(m+2)}{\Gamma(m+1)} = (m+1)\beta^{-1}. \tag{26}$$

Finding the mean of equation (25) from the MGF, in one step, shows

$$E(\mathbf{x}) = M'_{\mathbf{x}}(0) = \left[\frac{d\left(\int_0^{\infty} e^{tx} f(x)dx\right)}{dt}\right]_{t=0} = (m+1)\beta^{-1} \ , \tag{27}$$

which is equivalent to the result provided by the Moi formula method.

The relationship between the Moi MGF and the standard MGF can be shown for higher moments. To show this, consider the MGF and from it the jth moment. By definition, the jth moment based on an MGF for the density in equation (12) is

$$E(\mathbf{x}^j) = \left[\frac{d^j\left(\int_0^{\infty} e^{tx} f(x)dx\right)}{dt}\right]_{t=0} = \left[\int_0^{\infty} \frac{d^j f(x)e^{tx}}{dt} dx\right]_{t=0} = \int_0^{\infty} x^j f(x)dx \ ,$$

which leads to the same definition of the jth moment of the current approach given in equation (15).

To further exemplify the moments equations of this section, moments for the densities are listed in table 1, and moments functions for the selected univariate densities are listed in table 2.

Table 2. Moments Functions for Selected Univariate Densities

Density	PDF*	Moments Function†
Exponential	$\lambda e^{-\lambda x}$	$j\lambda^{-j}\Gamma(j)$
Gamma	$\frac{\lambda^{\mu}}{\Gamma(\mu)}x^{\mu}e^{-\lambda x}$	$\lambda^{-j}\frac{\Gamma(\mu+j)}{\Gamma(\mu)}$
Chi-Square	$\frac{x^{\frac{\nu}{2}-1}}{\Gamma\left(\frac{\nu}{2}\right)2^{\frac{\nu}{2}}}e^{-\frac{x}{2}}$	$\left(\frac{1}{2}\right)^{-j}\frac{\Gamma\left(\frac{\nu}{2}+j\right)}{\Gamma\left(\frac{\nu}{2}\right)}$
Rayleigh	$2axe^{-ax^2}$	$\frac{j}{2}a^{-\frac{j}{2}}\Gamma\left(\frac{j}{2}\right)$
Gamma-Poisson	$\frac{d(c\lambda)^m}{\Gamma(m)}x^{md-1}e^{-c\lambda x^d}$	$(c\lambda)^{\frac{-j}{d}}\frac{\Gamma\left(m+\frac{j}{d}\right)}{\Gamma(m)}$
Weibull	$abx^{b-1}e^{-ax^b}$	$\frac{j}{b}a^{\frac{-j}{b}}\Gamma\left(\frac{j}{b}\right)$
Maxwell	$\sqrt{2/\pi}\,x^2e^{\frac{-x^2}{2}}$	$\left(\frac{1}{2}\right)^{-j/2}\frac{\Gamma\left(\frac{3+j}{2}\right)}{\sqrt{\pi/2}}$

NOTE: The moments in this table exist on the general interval $-\infty < x < \infty$ and $0 < x < \infty$ is a special case.

* Calculated from equation (12). † Calculated from equation (15).

EXAMPLES OF THE METHOD

For the class of exponential functions under consideration, the PDF can be obtained by substituting coefficients and exponents into equation (12). From this function, it has been shown that the finite statistical moments can be obtained from equation (15). In particular, the mean of the distribution can be found by using equation (16), and the variance is obtained by using equation (17) for the second moment in conjunction with the first moment. In this section, several hypothetical functions are selected for demonstration purposes. The method is illustrated by the following examples.

FIRST EXAMPLE

Assume a researcher has reason to believe that certain data conform to the following structure:

$$g(x) = ax^2 e^{-ax^2},\ a \neq 0,\ 0 < x < \infty . \tag{28}$$

The value for the parameter is a known quantity in practice.

Because it has been many years since this engineering mathematician has studied mathematical statistics, it is decided to turn to the Moi MGF method for answers.

The parameters of the Moi model here are $\alpha = \beta = a;\ m = n = 2;\ \gamma = 3/2$. By using equations (11) and (12), the normalized density for the function in equation (28) can be obtained in one step. The normalizing constant is

$$\begin{aligned} c &= \frac{n\beta^{\gamma}}{\alpha\Gamma(\gamma)} , \\ &= \frac{2(a)^{\frac{3}{2}}}{a\Gamma\left(\frac{3}{2}\right)} , \\ &= \frac{2a^{\frac{1}{2}}}{\frac{\sqrt{\pi}}{2}} , \\ &= 4\sqrt{\frac{a}{\pi}} . \end{aligned} \tag{29}$$

Substituting the parameters n, β, and γ into equation (12) for the PDF $f(x)$, one obtains

$$f(x) = cg(x) ,$$

$$= \frac{n\beta^{\gamma} x^{m}}{\Gamma(\gamma)} e^{-\beta x^{n}} , \tag{30}$$

$$= 4\sqrt{\frac{a}{\pi}} a x^{2} e^{-a x^{2}} .$$

To find the statistical moments of the PDF, the general moments are obtained from equation (15):

$$E(\mathbf{x}^{j}) = \beta^{-j/n} \frac{\Gamma\left(\frac{m+j+1}{n}\right)}{\Gamma\left(\frac{m+1}{n}\right)} ,$$

$$= a^{-j/2} \frac{\Gamma\left(\frac{2+j+1}{2}\right)}{\Gamma\left(\frac{2+1}{2}\right)} ,$$

$$= a^{-j/2} \frac{\Gamma\left(\frac{3+j}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} . \tag{31}$$

Now, to obtain the mean, $j = 1$, and

$$E(\mathbf{x}) = \frac{a^{-\frac{1}{2}} \Gamma(2)}{\sqrt{\pi} / 2} ,$$

$$= \frac{2}{\sqrt{a\pi}} . \tag{32}$$

The second moment is now obtained:

$$E(\mathbf{x}^2) = \beta^{-2/2}\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{3}{2}\right)},$$

$$= a^{-1}\frac{\Gamma\left(\frac{3}{2}+1\right)}{\Gamma\left(\frac{3}{2}\right)}, \tag{33}$$

$$= \frac{3}{2a} .$$

The variance, $\sigma^2 = E(\mathbf{x}^2) - [E(\mathbf{x})]^2$ is then readily given by

$$\sigma^2 = \frac{3}{2a} - \left(\frac{2}{\sqrt{a\pi}}\right)^2 ,$$

$$= \frac{1}{a}\left(\frac{3\pi - 8}{2\pi}\right) . \tag{34}$$

SECOND EXAMPLE

Consider the exponential function

$$g(x) = e^{-ax^2}, \tag{35}$$

with parameters $\alpha = 1;\ \beta = a;\ m = 0;\ n = 2;\ \gamma = 1/2$. In practice, the value for the parameter a will be a known quantity. To summarize the results, calculations show

$$c = \frac{2a^{1/2}}{\Gamma\left(\frac{1}{2}\right)} = 2\sqrt{\frac{a}{\pi}} , \qquad \text{from (11)}$$

$$f(x) = cg(x) = 2\sqrt{\frac{a}{\pi}}x^2 e^{-ax^2} , \qquad \text{from (12)}$$

$$E(\mathbf{x}) = \beta^{-1/2} \frac{\Gamma\left(\frac{0+1+1}{2}\right)}{\left(\frac{0+1}{2}\right)} = a^{-1/2} \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{a\pi}} \ , \qquad \text{from (16)}$$

$$E(\mathbf{x}) = \beta^{-2/2} \frac{\Gamma\left(\frac{0+2+1}{2}\right)}{\left(\frac{0+1}{2}\right)} = \frac{a^{-1}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{2a} \ , \qquad \text{from (18)}$$

$$\sigma^2 = \frac{1}{2a} - \left(\frac{1}{\sqrt{a\pi}}\right)^2 = \frac{2-\pi}{4a} \ . \qquad \text{from (17)}$$

THIRD EXAMPLE

The third example will lead to the Rayleigh density. The exponential function of interest is

$$g(x) = axe^{-ax^2} \ , \tag{37}$$

where the identified parameters are $\alpha = \beta = a$; $m = 1$; $n = 2$; $\gamma = 1$. Applying the method, the results are summarized in the following way:

$$c = \frac{n\beta^{\gamma}}{\alpha\Gamma(\gamma)} = 2 \ , \qquad \text{from (11)}$$

$$f(x) = cg(x) = 2ax^2e^{-ax^2} \ , \qquad \text{from (12)}$$

$$E(\mathbf{x}) = \beta^{1/2} \frac{\Gamma\frac{m+1+1}{n}}{\Gamma\left(\frac{m+1}{n}\right)} = \frac{1}{2\sqrt{\frac{\pi}{a}}} \ , \qquad \text{from (16)}$$

$$E(\mathbf{x}^2) = \beta^{-2/2} \frac{\Gamma\left(\frac{m+2+1}{n}\right)}{\Gamma\left(\frac{m+1}{n}\right)} = \frac{1}{a} \ , \qquad \text{from (18)}$$

$$\sigma^2 = \frac{1}{a} - \frac{\pi}{4a} = \frac{4-\pi}{4a} \ . \qquad \text{from (17)} \tag{38}$$

FOURTH EXAMPLE

The last examples relate to underwater acoustics. A deterministic acoustic signal propagating through random media becomes a stochastic signal. If the media is strongly random, or the acoustic path through a weakly random media is long, the acoustic signal reaches a saturated state. The probability density of the signal intensity in the saturated state is exponential:

$$p(I) = (1/<I>)\exp(-1/<I>),$$

where

$$I(\mathbf{x},t,s) = \left| f(\mathbf{x},t,\omega)/f_0(\mathbf{x},\omega) \right|^2,$$

$<I>$ is the mean of the intensity, f is the amplitude of the signal at the receiver, f_0 is the amplitude of the signal at the source, $\mathbf{x}$ is a positional vector of the receiver, t is the time, and ω is source frequency.

Notice the Moi parameters α, m, n, β, γ in table 1 are defined as $1/<I>$, 0, 1, $<I>$, 1, respectively, for the exponential PDF.

As a related example, the PDF for the amplitude of a random signal that has achieved saturation is Rayleigh:

$$p(A) = 2A\exp(-A^2),$$

where

$$A = f(\mathbf{x},t,\omega)/f_0(\mathbf{x},\omega).$$

This region is also characterized by the real and imaginary parts of the complex signal having Gaussian PDF.

Notice that the Moi parameters α, m, n, β, γ in table 1 are defined $2A$, 1, 2, A, 1, respectively, for the Rayleigh PDF (see reference 11).

HYPOTHESIS TESTING

Once the moments have been derived from a density function, exact distributional probabilities can be calculated and hypotheses may be tested. For example, for the density $f(x) = 4\sqrt{\frac{a}{\pi}}ax^2e^{-ax^2}$, one may wish to know the probability that an observed value is ≤ 2; i.e., $P(\mathbf{x} \leq 2)$.. This involves evaluating

$$P(\mathbf{x} \leq 2) = \int_0^2 4\sqrt{\frac{a}{\pi}}ax^2e^{-ax^2}dx, \tag{39}$$

which represents an evaluation of the distribution function $F(x)$ $[0 \leq F(x) \leq 1]$ for the density function $f(x)$. The quantity a is, of course, a numerical value in a real situation and the exact probability can be computed. Although equation (39) represents a one-tailed hypothesis, a two-tailed hypothesis is equally valid; e.g., $P(1 \leq \mathbf{x} \leq 2)$ is the probability that an observed value lies between the interval of +1 to +2. Standard integration techniques such as integration by parts or Taylor-series expansions are available for such calculations as given in reference 10. For complicated formulas, an iterative reduction formula has been calculated in closed form for γ being a positive integer. This method was derived by O'Brien, Hammel, and Nguyen (references 12 and 13).

In addition, formal hypothesis testing protocols may be employed once the moments of the PDF are at hand. The central limit theorem or normal approximation formulas are typically of interest for evaluation of simple hypotheses. *Chebychev's Theorem*, which gives the probability of deviation from a mean regardless of the distribution, may also be of interest. The reference list may be consulted for standard works on mathematical statistics.

A possible application of testing hypotheses with the method developed in this report is statistical data analysis, in which, regression analysis techniques are used to model a data set. Figure 1 shows a set of data points (o) generated in some hypothetical time series that appears to conform to a negative exponential (or decay) function. The solid sloping-down arc labeled $g(\hat{x})$ is assumed to be an optimum least-squares solution derived for the discrete time series data.

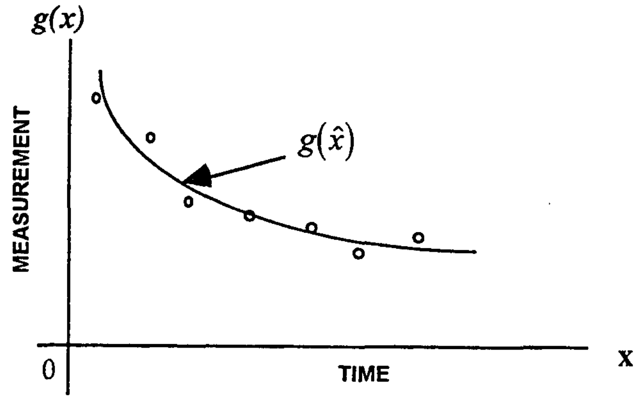

Figure 1. Hypothetical Data Conforming to Decay Function

The function $g(\hat{x})$ is obtained in the standard manner for exponential functions. First take the (natural) logarithm of the modeling basis (exponential) function $y = g(x) = \alpha x^m e^{-\beta x^n}$:

$$\log[g(x)] = \log\left[\alpha x^m e^{-\beta x^n}\right] = \log\alpha + m\log x - \beta x^n.$$

Because the term βx^n is nonlinearizable, a nonlinear approach must be taken. This approach consists of performing the regression analysis on $g(x)$, with the nonlinear parameter n *set* to a specific integer value, n = 1, 2, 3, etc. Each [x,y] observation in the sample is indexed with the subscript i, where i run from 1 to p. Then construct the least-squares criterion for a class of solutions:

$$f_n(\alpha,m,\beta,n) = \sum_{i=1}^{p}\left[\log y_i - \log\alpha - m\log x_i + \beta x_i^n\right]^2 \rightarrow \min.$$

The nonlinearity of the problem is apparent if one tries to take the partial derivative of $f_n(\alpha,m,\beta,n)$, with respect to n,

$$\frac{\partial\left(\sum_{i=1}^{p}\left[\log y_1 - \log\alpha - m\log x_i - \beta x_i^n\right]^2\right)}{\partial n}.$$

To circumvent this problem, a class of linear regression solutions is generated by fixing the value for n. That is, the first solution set is called $f_1(\alpha_1,m_1,\beta_1,1)$ because it is assumed that the parameter n = 1 in the least-squares regression solution. Next, we find the least-squares solution set member $f_2(\alpha_2,m_2,\beta_2,2)$ by setting n = 2. In this manner, a set of solutions is generated by systematically varying the value for the parameter n. Of the entire set of least-square solutions, the preferred solution set, say with $n = k$, is called $f_k(\alpha_k,m_k,\beta_k,k)$ and is the set that provides the smallest residual error (root-mean-square value). The solution $g(\hat{x})$ or $g_k(\hat{x}) = \hat{\alpha}\hat{x}e^{-\hat{\beta}\hat{x}^{\hat{n}}}$ is then obtained by taking antilog values of the parameter estimate set $\left[\hat{\alpha}_k,\hat{m}_k,\hat{\beta}_k\hat{k}\right]$, the empirical least-squares solution to the data generated in the time series. From $g_k(\hat{x})$, one could find the quantities earlier derived (the probablity density function and the moments).

SUMMARY

To summarize the results of this work, figure 2 shows the method steps for one-dimensional continuous distributions. To give a verbal description of the method, if an exponential function is of the form

$$g(x) = \alpha x^m e^{-\beta x^n}, \; 0 < x < \infty , \tag{40}$$

where $\alpha,\ m,\ n \neq 0,\ \beta > 0,$ and $\gamma = n^{-1}(m+1) > 0,$ then, to form a PDF, compute a normalizing constant c

$$c = \frac{n\beta^\gamma}{\alpha\Gamma(\gamma)} , \tag{41}$$

which provides the PDF,

$$f(x) = cg(x) = \frac{n\beta^\gamma x^m}{\Gamma(\gamma)} e^{-\beta x^n} , \tag{42}$$

and the jth statistical moment,

$$E(\mathbf{x}^j) = \beta^{-j/n} \frac{\Gamma\left(\gamma + \frac{j}{n}\right)}{\Gamma(\gamma)} = \beta^{-j/n} \frac{\Gamma\left(\frac{m+j+1}{n}\right)}{\Gamma\left(\frac{m+1}{n}\right)} , \tag{43}$$

such that the mean (first moment) is

$$E(\mathbf{x}) = \beta^{-1/n} \frac{\Gamma\left(\frac{m+2}{n}\right)}{\Gamma\left(\frac{m+1}{n}\right)} , \tag{44}$$

and the variance is

$$\sigma^2 = E\left(\mathbf{x}^2\right) - \left[E(\mathbf{x})\right]^2 , \tag{45}$$

where the second moment $E\left(\mathbf{x}^2\right)$ is computed from

$$E\left(\mathbf{x}^2\right) = \beta^{-2/n} \frac{\Gamma\left(\frac{m+3}{n}\right)}{\Gamma\left(\frac{m+1}{n}\right)} . \tag{46}$$

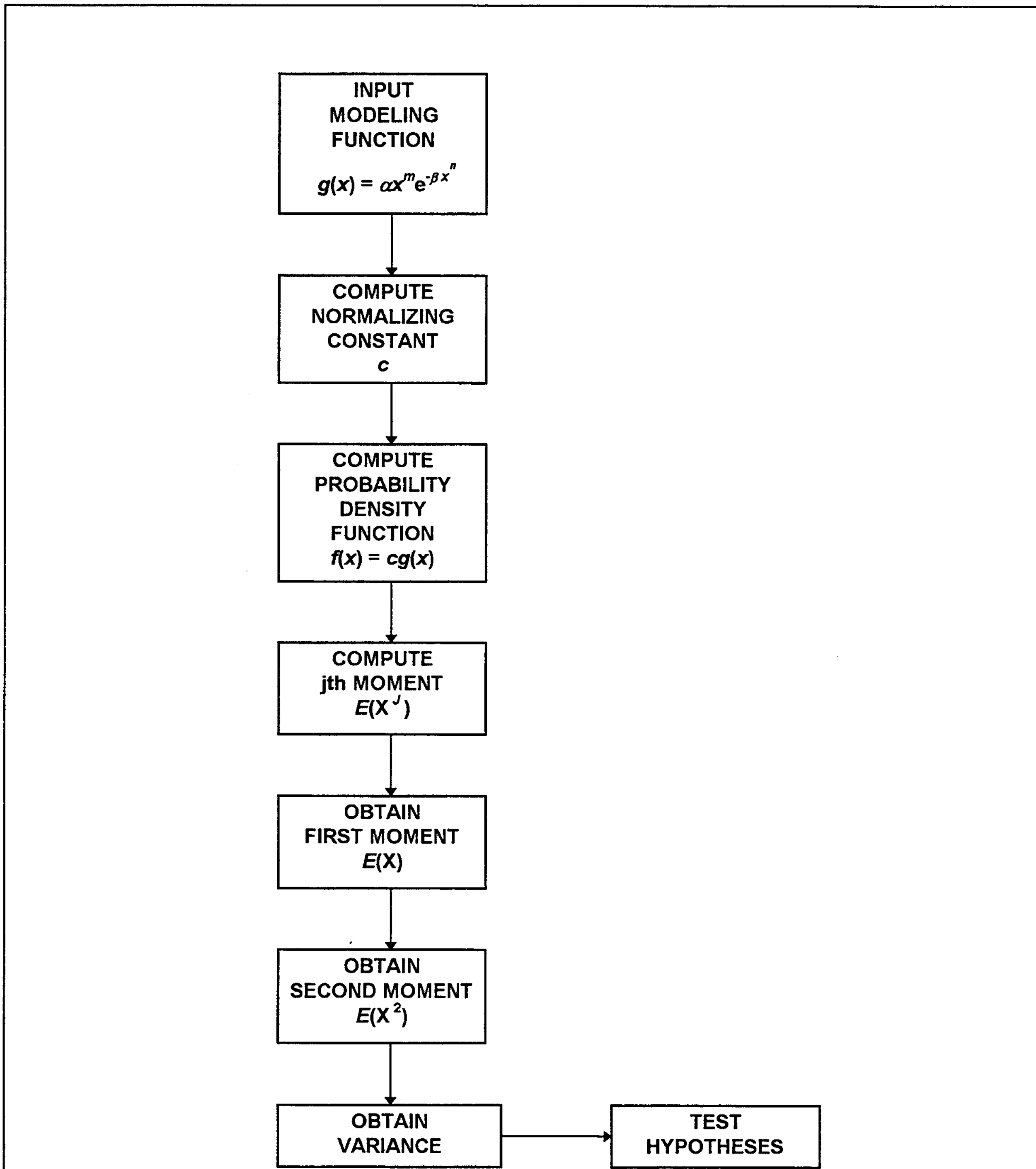

Figure 2. Summary of Moi Method Steps

The authors of this report have derived a general solution on $(-\infty < x < \infty)$. The equivalence between the current approach and moment-generating functions was shown generally. Other information was provided on testing statistical hypotheses from the central limit theorem and approximation formulas.

Finally, a generalization of the method of this report to multivariate distributions was given in a separate appendix. There it was shown that the joint PDF for n independent random variables is the product of individual densities of the form given in equation (42):

$$f(x_1,x_2,\ldots,x_n) = \prod_{i=1}^{n} f(x_i) \tag{47}$$

and the jth multivariate moments are obtained by forming the product of individual moments for each random variable

$$E\left(\boldsymbol{y}^j\right) = \prod_{i=1}^{n} E\left(\mathbf{x}_i^j\right). \tag{48}$$

where $\boldsymbol{y} = \sum_i \mathbf{x}_i$.

CONCLUSIONS

This report has presented a simple substitution method for finding a probability density function (PDF) and its statistical moments for an arbitrary exponential function of the form

$$g(x) = \alpha x^m e^{-\beta x^n},\ 0 < x < \infty,$$

where α, $\beta > 0$, m, n are any real constants in one-dimensional distributions and $g(x_1, x_2, \cdots, x_n)$ in the hyper plane. Such distributions arise frequently in signal processing, where x may represent time starting at $t = 0$ and g may represent some error or disturbance structure conforming to a decay function.

The general case, when $-\infty < x < \infty$, has subsequently been solved (see reference 4), the primary use of the method is to test statistically, hypotheses about the behavior of such functional forms once empirical least-squares methods have determined an applicable model derived from actual measurements (such as regression analyses of the time series in the ocean environment). That is to say, once the parameters α, β, m, and n are determined for a set of data measurements, the PDF-based mean and variance are determinable, and simple binary hypotheses may be tested, such as "the error term is essentially zero." Such tests are valid because they are based on the *central limit theorem* for sufficiently large n or *Chebychev's theorem,* both of which may be employed regardless of the underlying distribution.

REFERENCES

1. M. L. Graham, F. J. O'Brien, K. F. Gong, and S. E. Hammel, "Adaptive Statistical Filtering in Underwater Tracking with Correlated Measurement Noise," *Proceedings of the International Conference on Acoustics, Speech, and Signal Processing,* 1993.

2. F. J. O'Brien, C. T. Nguyen, and S. E. Hammel, "Signal Characterization Applications of Poisson Point Processes in the Ocean Environment," NUWC-NPT Technical Report 10, 412, Naval Undersea Warfare Center Division, Newport, RI, 1 November 1994.

3. P. J. Hoel, S. C. Port, and C. J. Stone, *Introduction to Probability Theory,* Houghton-Mifflin, Boston, 1971.

4. F. J. O'Brien, "An Extension of the Moi Formula," *Perceptual and Motor Skills,* in press, 1995.

5. J. E. Freund, *Mathematical Statistics, 2nd Edition,* Prentice-Hall, New Jersey, 1971.

6. F. J. O'Brien, S. E. Hammel, and C. T. Nguyen, "The Moi Formula," Accepted in I. S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series and Products,* Academic Press, New York, 1994.

7. F. J. O'Brien, S. E. Hammel, and C. T. Nguyen, "The Moi Formula for Improper Exponential Definite Integrals," *Perceptual and Motor Skills,* 79, 1994, pp. 1123-1127.

8. R. V. Hogg and A. T. Craig, *Introduction to Mathematical Statistics, 2nd Edition,* Macmillan, New York, 1969.

9. I. S. Sokolnikoff, *Advanced Calculus,* McGraw-Hill, New York, 1939.

10. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions,* Government Printing Office, Washington, DC, 1964.

11. S. M. Flatté, R. Dashen, W. H. Munk, K. M. Watson,. and F. Zachariasen, *Sound Transmission Through a Fluctuating Ocean,* Cambridge University Press, 1979.

12. F. J. O'Brien, S. E. Hammel, and C. T. Nguyen, "A General Exponential Integral Formula," *Perceptual and Motor Skills,* 79, 1994, pp. 1645-1646.

13. F. J. O'Brien, S. E. Hammel, and C. T. Nguyen, "Induction Proof for a General Exponential Integral Formula," *Perceptual and Motor Skills,* in press, 1995.

14. L. Bers, *Calculus, Volume 1,* Holt, Rinehart, and Winston, New York, 1969.

15. G. S. Carr, *Formulas and Theorems in Pure Mathematics,* Chelsea Publishing Company, New York, 1970.

APPENDIX A
DERIVATION OF THE MOI FORMULA

The improper exponential integral to be derived is

$$I = \int_0^\infty \alpha x^m e^{-\beta x^n} dx \;, \; 0 < x < \infty , \qquad \text{(A-1)}$$

where $\alpha,\ m,\ n \neq 0,\ \beta > 0,\ \gamma = n^{-1}(m+1) > 0.$

The integral I clearly exists since the integrand is a continuous function which is bounded by an integrable function (reference 9).

To evaluate I, set

$$s = \beta x^n, \text{ so that } x = \left(s\beta^{-1}\right)^{\frac{1}{n}},\ ds = n\beta x^{n-1} dx \;, \qquad \text{(A-2)}$$

and substituting into I

$$I = \alpha(n\beta)^{-1} \int_0^\infty x^{m+1-n} e^{-s} ds \;, \qquad \text{(A-3)}$$

so that substituting γ gives

$$I = \alpha(n\beta^{\gamma})^{-1} \int_0^\infty s^{\gamma-1} e^{-s} ds \;, \qquad \text{(A-4)}$$

where the integral is seen to be the gamma function $\Gamma(\gamma)$, so that

$$I = \left(a^{-1} n \beta^{\gamma}\right)^{-1} \Gamma(\gamma),$$

$$I = \frac{\alpha \Gamma\left(\frac{m+1}{n}\right)}{n\beta^{\left[(m+1)/n\right]}} \;, \qquad \text{(A-5)}$$

which completes the derivation.

APPENDIX B

INDEPENDENT PROOF THAT $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

In this appendix, an independent proof is given for the well-known relation between the gamma function and π, viz., $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. To do this, the derivation of the Moi formula is again outlined to make this appendix self-contained, which allows the authors to relate their version to standard approaches.

The gamma function is given as

$$\Gamma(x;\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx, \quad \alpha > 0 \ , \tag{B-1}$$

so that for $\alpha = \frac{1}{2}$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. The proof of this statement has been given in various places.

The calculus text by Bers (reference 14, pp. 402-403) gives one approach. Since

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \int_0^\infty 2\, e^{-x^2} dx = \sqrt{\pi} \ , \tag{B-2}$$

by transformation to a double integral in polar coordinates, then by substituting in equation (B-1) the quantities $x = y^2$, $dx = 2x^{1/2} dy$,

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty x^{-1/2} e^{-x} dx = \int_0^\infty 2e^{-y^2} dy = \sqrt{\pi} \ , \tag{B-3}$$

The proof by Carr (reference 15, theorem 2314) proceeds along similar lines. Other approaches exist (e.g., reference 9).

NEW PROOF

Our version involves first a generalization of an elementary exponential integral found in standard handbooks. Consider the following integral:

$$\int_0^\infty \alpha x^m e^{-\beta x^n} dx, \qquad 0 < x < \infty \ , \tag{B-4}$$

where ($\alpha \neq 0$, $\beta > 0$, m, $n \neq 0$) are real-valued constants. To evaluate equation (B-4), make the change of variable by setting $s = \beta x^n$, so that $x = \left(s\beta^{-1}\right)^{1/n}$, $ds = n\beta x^{n-1}dx$, and letting $\gamma = \dfrac{m+1}{n} > 0$, to get

$$\alpha\left(n\beta^\gamma\right)^{-1}\int_0^\infty s^{\gamma-1}e^{-s}ds, \quad s > 0 \ . \tag{B-5}$$

The integral in equation (B-5) is recognized to be a form of the gamma function, in (B-1):

$$\int_0^\infty s^{\gamma-1}e^{-s}ds = \Gamma(\gamma), \ \gamma > 0 \ .$$

Thus,

$$\int_0^\infty \alpha x^m e^{-\beta x^n} dx = \alpha \frac{\Gamma(\gamma)}{n\beta^\gamma} \ , \tag{B-6}$$

which is the Moi formula introduced earlier.

With this integral formula, consider the following

$$\int_0^\infty e^{-\beta x^n} dx,$$

which can be evaluated by equation (B-6) and simplified to

$$n\beta^{1/n}\int_0^\infty e^{-\beta x^n} dx = \Gamma\left(\frac{1}{n}\right). \tag{B-7}$$

With this structure, two of the known approaches to the proof can be connected. With $n = 2$, then

$$2\beta^{1/2}\int_0^\infty e^{-\beta x^2} dx = \Gamma\left(\frac{1}{2}\right), \tag{B-8}$$

so that if $\beta = 1$, then from equation (B-2)

$$2\int_0^\infty e^{-x^2} dx = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \tag{B-9}$$

To approach the proof another way, let $\beta = \frac{1}{2}$ in equation (B-8), giving

$$\frac{1}{\sqrt{2}}\int_0^\infty 2e^{-x^2/2} dx = \Gamma\left(\frac{1}{2}\right). \tag{B-10}$$

Now, textbooks on calculus and probability theory prove (via polar coordinate transformation) that the integral in equation (B-10) is related to the probability integral

$$\int_{-\infty}^\infty e^{-x^2/2} dx = \sqrt{2\pi} \tag{B-11}$$

or

$$\int_{-\infty}^\infty e^{-x^2/2} dx = \int_0^\infty 2e^{-x^2/2} dx = \sqrt{2\pi} \ . \tag{B-12}$$

Evaluating equation (B-11) now by the Moi formula of equation (B-6) produces

$$\int_{-\infty}^\infty e^{-x^2/2} dx = \int_0^\infty 2e^{-x^2/2} dx = \sqrt{2}\,\Gamma\left(\frac{1}{2}\right) \ . \tag{B-13}$$

and so equating (B-12) and (B-13) produces the desired result:

$$\frac{\sqrt{2\pi}}{\sqrt{2}} = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

The proof is now complete.

Other variations of the proof are possible with the Moi integral formula introduced in equation (B-6). For example, one could start with $\gamma = \frac{m+1}{n} = \frac{0+1}{2}$ in equation (B-6) and solving the resulting equation (B-8) by polar coordinates gives the desired result.

APPENDIX C
MULTIVARIATE DISTRIBUTIONS

For the multivariate case of n independent random variables, the authors considered functions of the type $\mathbf{y} = \mathbf{x}_1 + \mathbf{x}_2$, or the sum of two random variables, $\mathbf{x}_1$ and $\mathbf{x}_2 < x < \infty$.* In this appendix, the generalization of the univariate method is outlined. Reference 4 (chapter 6) is a good readable reference for the material of this appendix.

NOTATION

Whereas the authors represented a single random variable by $\mathbf{x}$ and defined the PDF $f(x)$ in terms of the single variable by the relation $f(x) = cg(x) = \dfrac{n\beta^{\gamma} x^{m}}{\Gamma(\gamma)} e^{-\beta x^{n}}$, and the general Moi moments relation by the function

$$E\left(\mathbf{x}^{j}\right) = \beta^{-j/n} \frac{\Gamma\left(\gamma + \frac{j}{n}\right)}{\Gamma(\gamma)} = \beta^{-j/n} \frac{\Gamma\left(\frac{m+j+1}{n}\right)}{\Gamma\left(\frac{m+1}{n}\right)},$$

each random variable must be indexed for clarity. Let each occurrence of a random variable $\mathbf{x}$ be denoted now $\mathbf{x}$ and let all relations previously defined ($g(x), f(x), E\left(\mathbf{x}^{j}\right)$ etc.) now be subscripted appropriately to indicate a single occurrence of an independent random variable.

For example, let a univariate density for a random variable $\mathbf{x}_1$ be denoted

$$f(x_1) = \frac{n_1 \beta_1^{\gamma_1} x_1^{m_1}}{\Gamma(\gamma_1)} e^{-\beta_1 x_1^{n_1}},$$

and let the Moi moments relation be denoted

$$E(\mathbf{x}_1^{j}) = \beta_1^{-j/n_1} \frac{\Gamma\left(\gamma_1 + \frac{j}{n_1}\right)}{\Gamma(\gamma_1)} = \beta_1^{-j/n_1} \frac{\Gamma\left(\frac{m_1 + j + 1}{n_1}\right)}{\Gamma\left(\frac{m_1 + 1}{n_1}\right)}.$$

---

* The case for a linear combination is a straightforward extension; $\mathbf{z} = \mathbf{a}_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots + \mathbf{a}_n\mathbf{x}_n$, where $\mathbf{a}_j$ is some assumed constant.

where

$$\gamma_1 = \frac{m_1 + 1}{n_1} \ .$$

If $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n$ are independent random variables with Moi moment functions, $E(\mathbf{x}_1^j), E(\mathbf{x}_2^j), E(\mathbf{x}_3^j), \ldots, E(\mathbf{x}_n^j)$ and $\mathbf{y} = \sum_i \mathbf{x}_i$, then

$$E(\mathbf{y}^j) = \prod_{i=1}^{n} E(\mathbf{x}_i^j),$$

where

$$E(\mathbf{x}_i^j) = \beta_i^{-j/n_i} \frac{\Gamma\left(\gamma_i + \frac{j}{n_i}\right)}{\Gamma(\gamma_i)}, \quad \gamma_i = \frac{m_i + 1}{n_i} \ . \tag{C-1}$$

PROOF OUTLINE

Because the random variables $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n$ are independent, and thus, the joint density $f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i)$ by conventional definition, then the following is true:

$$E(\mathbf{y}^j) = \int_0^\infty \int_0^\infty \ldots \int_0^\infty x_1^j f(x_1)\, x_2^j f(x_2) \ldots x_n^j f(x_n)\, dx_1 dx_2 \ldots dx_n \,,$$

$$E(\mathbf{y}^j) = \int_0^\infty x_1^j f(x_1) dx_1 \int_0^\infty x_2^j f(x_2) dx_2 \ldots \int_0^\infty x_n^j f(x_n)\, dx_n \,,$$

$$E(\mathbf{y}^j) = \beta_1^{\frac{-j}{n_1}} \frac{\Gamma\left(\gamma_1 + \frac{j}{n_1}\right)}{\Gamma(\gamma_1)} \bullet \beta_2^{\frac{-j}{n_2}} \frac{\Gamma\left(\gamma_2 + \frac{j}{n_2}\right)}{\Gamma(\gamma_2)} \bullet \ldots \bullet \beta_n^{\frac{-j}{n_n}} \frac{\Gamma\left(\gamma_n + \frac{j}{n_n}\right)}{\Gamma(\gamma_n)}$$

$$E(y^j) = \prod_{i=1}^{n} E(x_i^j) \ . \tag{C-2}$$

Hence, the joint PDF is computed from the following relation:

$$f(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} \frac{n_i b_i^{g_i} x_i^{m_i}}{\Gamma(\gamma_i)} e^{-b_i x_i^{n_i}} ,$$

where

$$\prod_{i=1}^{n} \int_0^{\infty} f(x_i) dx_i = 1 . \tag{C-3}$$

In conclusion, the case of n independent variables involves taking certain products of the univariate relations.

EXAMPLE

Assume that data are modeled by a two-variable function of the form $g(x_1, x_2) = e^{-(3x_1+2x_2)}, (x_1, x_2) > 0$. Find the joint PDF and general moments for function g. The PDF is

$$f(x_1, x_2) = cg(x_1, x_2),$$

where c is the normalizing constant:

$$f(x_1, x_2) = \left(\prod_{i=1}^{i=2} \frac{n_i \beta_i^{\gamma_i} x^{m_i}}{\Gamma(\gamma_i)} \right) e^{-\sum_i \beta_i x_i^{n_i}} ,$$

$$f(x_1, x_2) = \left(\prod_{i=1}^{i=2} \frac{n_i \beta_i^{\gamma_i} x_i^{m_i}}{} \right) e^{\sum_i \beta_i x_i^{n_i}} ,$$

$$f(x_1, x_2) = \left(3x_1 e^{-3x_1}\right) \bullet \left(2x_2 e^{-2x_2}\right) ,$$

$$f(x_1, x_2) = 6e^{-(3x_1+2x_2)}, \quad (x_1, x_2) > 0 ,$$

so that

$$\int_0^{\infty} f(x_1) dx_1 \bullet \int_0^{\infty} f(x_2) dx_2 = 1 . \tag{C-4}$$

The general moments function is derived as follows:

$$E\left(\mathbf{y}^j\right)=\prod_{i=1}^{2} E\left(\mathbf{x}_i^j\right) ,$$

$$E\left(\mathbf{y}^j\right)=\frac{3^{-j}\Gamma(1+j)}{\Gamma(1)}\bullet\frac{2^{-j}\Gamma(1+j)}{\Gamma(1)} ,$$

$$E\left(\mathbf{y}^j\right)=6^{-j} j^2\Gamma(j) ,$$

so that the mean is $\frac{1}{6}$ and the variance is

$$\frac{4}{36}-\frac{1}{36}=\frac{1}{12}. \qquad \text{(C-5)}$$

The generalization of the multivariate methods to $(-\infty,\infty)$ exponentials is a straightforward exercise and follows the integral derived in reference 4.

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